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More Inequalities for Critical Exponents

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A variety of rigorous inequalities for critical exponents is proved. Most notable is the low-temperature Josephson inequality $d\nu' \geq \gamma' + 2\beta \geq 2 - \alpha'$. Others are $1 \leq \gamma' \leq 1 + \nu'_\phi$, $1 \leq \zeta \leq 1 + \delta\mu_\phi$, $\delta \geq 1$, $d\mu_\phi \geq 1 + 1/\delta$ (for $\phi \geq d$), $d\nu'_\phi \geq \Delta'_3 + \beta$ (for $\phi \geq d$), $\Delta_4 \geq \gamma$, and $\Delta_{2m} \leq \Delta_{2m+2}$ (for $m \geq 2$). The hypotheses vary; all inequalities are true for the spin-1/2 Ising model with nearest-neighbor ferromagnetic pair interactions.

KEY WORDS: Critical exponents; critical-exponent inequalities; correlation inequalities; Josephson inequality.

1. INTRODUCTION

Almost two decades ago, Rushbrooke[(1)] initiated what has by now become a minor industry: the rigorous demonstration of inequalities relating the critical exponents associated with systems undergoing a phase transition. Since Rushbrooke's seminal work, numerous such inequalities have been proven;[2] so there is probably little harm in adding a few more to the list. At any rate, that is what I will do in this paper.

At least one of the inequalities proven here is of genuine physical significance: I give the first rigorous proof (to my knowledge) of the low-temperature Josephson[(5)] inequality $d\nu' \geq 2 - \alpha'$. (Actually, I prove the slightly stronger result $d\nu' \geq \gamma' + 2\beta$.) For the rest, I consider most of the inequalities proven here to be more amusing than useful. This statement merits a brief explanation.

I consider the methods of proof to be rather amusing, for three

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[2] For some reviews, see Stanley[(2)] and Fisher.[(3,4)]

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reasons:

(1) While the use of correlation inequalities in proving critical-exponent inequalities has a long and honorable history (for a beautiful early example, see Fisher[4]), I here carry this tradition to its logical extreme, by using correlation inequalities in ways they were surely not meant to be used. My method of proof can only be described as "squeezing everything in sight out of every correlation inequality in sight," without regard to physical interpretation. As a result, I get what I deserve: some of my inequalities merit little more comment than the casual remark that they are true.

(2) The proofs typically begin by considering a quanitity which is apparently unrelated to the inequality at hand, for example, by considering the temperature derivative of the magnetization while attempting to prove an inequality involving only the susceptibility. What invariably happens is that the "wrong" quantities cancel out at the final step, leaving only the "right" quantities.

(3) Many existing proofs of critical-exponent inequalities work only for the high-temperature exponents, not for the low-temperature or critical-isotherm ones: these proofs fail in the presence of nonzero magnetization. An example is Fisher's proof[4] of the inequality $\gamma \leqslant (2-\eta)\nu_\phi$; while the corresponding low-temperature inequality $\gamma' \leqslant (2-\eta)\nu'_\phi$ is also expected to be true, it has not yet been proven. By contrast, most of the proofs given here work *only* in the presence of nonzero magnetization: in zero magnetization, they reduce to such profundities as $0 \geqslant 0$.

Critical-exponent inequalities can be divided usefully into five classes:

(1) Those that become equality under a thermodynamic homogeneity (scaling) hypothesis, e.g., Rushbrooke's $\alpha' + 2\beta + \gamma' \geqslant 2$;

(2) Those that become equality under a length scaling hypothesis, e.g., Fisher's $\gamma \leqslant (2-\eta)\nu_\phi$;

(3) Those that become equality under the hyperscaling hypothesis,[3] e.g., the Buckingham–Gunton[4,8,9] inequality $2-\eta \leqslant d(\delta-1)/(\delta+1)$;

(4) Those that become equality when the exponents take their classical (mean-field) values, e.g., the inequality $\gamma \geqslant 1$ due to Glimm and Jaffe;[10,11],[4] and

(5) Those that become equality under no reasonable condition.

Previously proved inequalities have usually been of the first four types (but see, e.g., Ref. 14 for an exception). It is a consequence of the present

[3] For the distinctions between the three successively stronger scaling hypotheses, see Hankey and Stanley[6] and Fisher.[7]

[4] It is worth remarking that the inequality $\alpha \leqslant \max[0, (2-d/2)\gamma]$, due to the present author,[12] becomes equality for the Gaussian model, a slight generalization of mean-field theory.[13]

Table I. Inequalities Proven in This Paper

Quantity bounded	Type of bound	Assumptions	Region	Critical-exponent inequality	Conditions under which equality attained	Section of this paper
$\sum_{j,k} \alpha_{jk} \langle \varphi_i ; \varphi_j \varphi_k \rangle$	Upper	GHS	$J > J_c$	$\gamma' \geqslant 1$	Classical	3
			$J = J_c$	$\zeta \geqslant 1$	Classical	3
	Lower	Nearest-neighbor ferromagnetic pair interaction	$J > J_c$	$\gamma' \leqslant 1 + \nu'_\phi$	——	3
			$J = J_c$	$\zeta \leqslant 1 + \delta \mu_\phi$	——	3
$\sum_{j,k} \langle \varphi_i ; \varphi_j ; \varphi_k \rangle$	Upper	GHS	$J = J_c$	$\delta \geqslant 1$	——	4
	Lower	Ginibre or New Lebowitz	$J > J_c$	$d\nu'_\phi \geqslant \Delta'_3 + \beta \quad (\phi \geqslant d)$	Hyperscaling	4
			$J = J_c$	$d\nu_\phi \geqslant 1 + 1/\delta \quad (\phi \geqslant d)$	Hyperscaling	4
$\sum_{j,k,l} \alpha_{kl} \langle \varphi_i ; \varphi_j ; \varphi_k \varphi_l \rangle$	Lower	New Lebowitz (or Ginibre)	$J > J_c$	$d\nu'_\phi \geqslant \gamma' + 2\beta \; (d/\phi)(\gamma' - 1) \; (\phi \geqslant d) \; \leqslant 2 - \alpha' \, (d/\phi)(\gamma' - 1)$	——	5
		New Lebowitz (or Ginibre) and reflection positivity	$J > J_c$	$d\nu' \geqslant \gamma' + 2\beta \geqslant 2 - \alpha'$	Hyperscaling	5
$\bar{u}_4$	Upper	Strange GHS (spin-1/2 only)	$J < J_c$	$\Delta_4 \geqslant \gamma$	——	6
$\lvert \bar{u}_{2m} \rvert$	Upper	Lee–Yang	$J < J_c$	$\Delta_{2m} \leqslant \Delta_{2m+2} \quad (m \geqslant 2)$	Thermodynamic homogeneity	6

paper's perverse use of correlation inequalities that several of the inequalities proven here are of type (5)—well, so be it.

The main results of this paper are summarized in Table I. Different inequalities require different assumptions; however, all inequalities are true for the spin-1/2 Ising model with nearest-neighbor ferromagnetic pair interactions (in dimension $d \geqslant 2$). Most are also true for the φ^4 lattice field theory.(15,16) The most interesting inequalities, to my mind, are those which become equality under hyperscaling. The hyperscaling conjecture,(7) whose validity is unknown even for the three-dimensional Ising model,(17–21) is closely related to the nontriviality of the continuum limit in quantum field theory.(18,22)

2. NOTATION AND PRELIMINARIES

In this paper, we shall consider a lattice system of classical one-component (real-valued) spins φ_i, where i ranges over the points of a d-dimensional lattice $\mathcal{L}$ (usually $\mathcal{L} = \mathbb{Z}^d$). Such a model can be specified by an *interaction* $\{\Phi_X\}_{X \subset \mathcal{L}}$ and a family of *a priori single-spin measures* $\{\nu_i\}_{i \in \mathcal{L}}$. Here X ranges over the nonempty finite subsets of $\mathcal{L}$, and the Φ_X are functions of the spins $\{\varphi_i\}_{i \in X}$. Each ν_i is a probability measure on the real line. *Formally* the Hamiltonian is $\sum_X \Phi_X$; of course, this actually makes sense only in finite volume.

To make this precise, let $b \in \mathbb{R}^{\mathcal{L}}$ be a spin configuration. For reasonable $\{\Phi_X\}$, $\{\nu_i\}$, and b, one can define the *Gibbs measure in (finite) volume* Λ *with boundary condition* b:

$$d\mu^{\Phi}_{\Lambda,b}(\varphi) = \left(Z^{\Phi}_{\Lambda,b}\right)^{-1} \exp\left[-\sum_{X \cap \Lambda \neq \varnothing} \Phi_X(\varphi)\right] \prod_{i \in \Lambda} d\nu_i(\varphi_i) \prod_{i \in \mathcal{L} \setminus \Lambda} \delta(\varphi_i - b_i)\, d\varphi_i \tag{2.1}$$

Here the partition function $Z^{\Phi}_{\Lambda,b}$ is defined so as to make $\mu^{\Phi}_{\Lambda,b}$ a probability measure. We write $\langle \cdots \rangle^{\Phi}_{\Lambda,b}$ for the expectation with respect to $\mu^{\Phi}_{\Lambda,b}$, and omit the labels Φ, Λ, and/or b if they are clear from the context.

More particularly, we shall consider the following special case: Let $A = \{A_i\}_{i \in \mathcal{L}}$ be a family of nonnegative integers, only a finite number of which are nonzero; and define $\varphi^A = \prod_{i \in \mathcal{L}} \varphi_i^{A_i}$, $\operatorname{supp} A = \{i \in \mathcal{L} : A_i \neq 0\}$, and $|A| = \sum_{i \in \mathcal{L}} A_i$. Then let

$$\Phi_X(\varphi) = -\sum_{A\,:\, \operatorname{supp} A = X} J_A \varphi^A \tag{2.2}$$

This is a model with polynomial interaction. To ensure the finiteness of the measure (2.1), we shall assume that $\int \exp(a|\varphi|^D)\, d\nu_i(\varphi) < \infty$ for all a and i, where $D = \max_{A\,:\,J_A \neq 0}|A|$ is the maximal degree of interaction.[5] If the interaction coefficients J_A are all nonnegative and the single-spin measures are all even, then the model is called *ferromagnetic*.

For reasonable $\{\Phi_X\}$, $\{\nu_i\}$, and b, it is possible to show that the measures $\mu^{\Phi}_{\Lambda,b}$ converge (in a suitable topology) as the volume Λ increases to encompass all of $\mathfrak{L}$.[6] This limiting measure is called an *infinite-volume Gibbs state* for the given interaction; it need not be unique, since it can depend on the boundary condition b. We shall restrict ourselves to translation-invariant interactions ($J_A = J_{A+i}$ for all $i \in \mathfrak{L} = \mathbb{Z}^d$) and to boundary conditions which yield a translation-invariant, ergodic ("pure phase") infinite-volume limit.

Ferromagnetic models obey a number of well-known correlation inequalities. These are first proven in finite volume (with suitable boundary conditions); they then carry over immediately to the infinite-volume limit. The following correlation inequalities will be used in the present work:

(1) Griffiths–Kelly–Sherman (GKS) inequalities I and II[28,29]: Assume that the model is ferromagnetic and $b_i \geqslant 0$ for all i. Then

$$\text{(I)}\quad \langle \varphi^A \rangle \geqslant 0 \qquad \text{for all } A$$

and

$$\text{(II)}\quad \langle \varphi^A; \varphi^B \rangle \equiv \langle \varphi^A \varphi^B \rangle - \langle \varphi^A \rangle \langle \varphi^B \rangle \geqslant 0 \qquad \text{for all } A, B.$$

(2) Ginibre inequality[28,30]: Assume that the model is ferromagnetic and $b_i \geqslant 0$ for all i. Let $\{\varphi_i\}$ and $\{\varphi_i'\}$ be independent identical copies of the given model, and define $q_i = 2^{-1/2}(\varphi_i - \varphi_i')$, $t_i = 2^{-1/2}(\varphi_i + \varphi_i')$, $q^A = \prod_{i \in \mathfrak{L}} q_i^{A_i}$ and $t^B = \prod_{i \in \mathfrak{L}} t_i^{B_i}$. Then $\langle q^A t^B \rangle \geqslant 0$ for all A, B.

(3) New Lebowitz inequality[31,32]: Assume that the model is ferromagnetic and $b_i \geqslant 0$ for all i. Then

$$\langle \varphi^A; \varphi^B \varphi^C \rangle \geqslant |\langle \varphi^B \rangle \langle \varphi^A; \varphi^C \rangle - \langle \varphi^C \rangle \langle \varphi^A; \varphi^B \rangle|$$

(This obviously strengthens GKS II.) This (together with its permutations) can also be written in the form

$$\langle \varphi^A; \varphi^B; \varphi^C \rangle \geqslant -2\min\left[\langle \varphi^A \rangle \langle \varphi^B; \varphi^C \rangle, \langle \varphi^B \rangle \langle \varphi^A; \varphi^C \rangle, \langle \varphi^C \rangle \langle \varphi^A; \varphi^B \rangle\right]$$

where

$$\langle \varphi^A; \varphi^B; \varphi^C \rangle \equiv \langle \varphi^A \varphi^B \varphi^C \rangle - \langle \varphi^A \rangle \langle \varphi^B \varphi^C \rangle - \langle \varphi^B \rangle \langle \varphi^A \varphi^C \rangle - \langle \varphi^C \rangle \langle \varphi^A \varphi^B \rangle + 2\langle \varphi^A \rangle \langle \varphi^B \rangle \langle \varphi^C \rangle$$

[5] This condition is slightly more stringent than is really necessary.

[6] More precisely, it can be shown for very general interactions and boundary conditions that a *subsequence* $\mu^{\Phi}_{\Lambda_n,b}$ converges.[23–25] For *ferromagnetic* models and boundary condition $b = 0$, the full net $\mu^{\Phi}_{\Lambda,b}$ converges, by an argument using the GKS inequalities.[26,27] For a somewhat different class of models, with "+" or "−" boundary conditions, the full net $\mu^{\Phi}_{\Lambda,b}$ converges, by an argument using the FKG[23,25] or GKS[26] inequalities.

(4) (Old) Lebowitz inequality[28,33–35]: Assume that the model is ferromagnetic, with $J_A = 0$ for $|A| > 2$, and $b_i \geqslant 0$ for all i. Assume further that each single-spin measure ν_i is one of the following[7]:

(a) $d\nu_i(\varphi) = \exp[-V(\varphi)]\,d\varphi$, where V is even and differentiable, with V' convex on $(0, \infty)$;

or

$$\text{(b)} \quad d\nu_i(\varphi) = \frac{1}{l+1} \sum_{j=0}^{l} \delta(\varphi - l + 2j)\, d\varphi \qquad (\text{spin-}l/2 \text{ model})$$

Then

$$\langle q^A q^B \rangle \geqslant \langle q^A \rangle \langle q^B \rangle$$
$$\langle t^A t^B \rangle \geqslant \langle t^A \rangle \langle t^B \rangle$$
$$\langle q^A t^B \rangle \leqslant \langle q^A \rangle \langle t^B \rangle$$

It is the last of these inequalities which is the most useful; among its consequences are

(4a) Griffiths–Hurst–Sherman (GHS) inequality[28,34,35]:

$$\begin{aligned} \langle \varphi_i ; \varphi_j ; \varphi_k \rangle \equiv \langle \varphi_i \varphi_j \varphi_k \rangle - \langle \varphi_i \rangle \langle \varphi_j \varphi_k \rangle - \langle \varphi_j \rangle \langle \varphi_i \varphi_k \rangle \langle \varphi_k \rangle \langle \varphi_i \varphi_j \rangle \\ + 2 \langle \varphi_i \rangle \langle \varphi_j \rangle \langle \varphi_k \rangle \leqslant 0 \end{aligned}$$

(4b) Lebowitz inequality for the four-point function[28,33,34]:

$$\begin{aligned} \langle \varphi_i \varphi_j \varphi_k \varphi_l \rangle - \langle \varphi_i \varphi_j \rangle \langle \varphi_k \varphi_l \rangle - \langle \varphi_i \varphi_k \rangle \langle \varphi_j \varphi_l \rangle \\ - \langle \varphi_i \varphi_l \rangle \langle \varphi_j \varphi_k \rangle + 2 \langle \varphi_i \rangle \langle \varphi_j \rangle \langle \varphi_k \rangle \langle \varphi_l \rangle \leqslant 0 \end{aligned}$$

In Section 3, we shall also use an inequality for the two-point function due to Schrader,[38] Messager and Miracle-Sole,[39] and Hegerfeldt.[40] In Section 6, we shall use some inequalities derived by Newman[41] from the Lee–Yang theorem, as well as a rather strange inequality due to GHS.[42] In the Appendix we shall use the FKG[29,43,44] and Gaussian[45–47] inequalities.

Most commonly, we shall consider a model with ferromagnetic pair interactions and a possible magnetic field:

$$J_A = \begin{cases} J_{ij} & \text{if } |A| = 2 \quad \text{and} \quad A_i = A_j = 1 \\ H & \text{if } |A| = 1 \\ 0 & \text{otherwise} \end{cases} \tag{2.3}$$

Here $H \geqslant 0$ and $J_{ij} = J_{ji} \geqslant 0$; moreover, J_{ij} depends only on $i - j$, and $\tilde{J} = \sum_j J_{ij} < \infty$. We shall generally fix the geometric structure of the pair

[7] Actually, somewhat more general ν_i are allowed, by virtue of the "analog system" method of Griffiths.[36,37]

interaction and vary only its strength, i.e.,

$$J_{ij} = J\alpha_{ij} \tag{2.4}$$

with fixed coefficients $\alpha_{ij} = \alpha_{ji} \geqslant 0$, and $\tilde{\alpha} = \sum_j \alpha_{ij} < \infty$. The quantity $T = 1/J$ can be considered as a "temperature." A typical example is the nearest-neighbor interaction, in which $\alpha_{ij} = 1$ if $|i-j| = 1$, $\alpha_{ij} = 0$ otherwise.

We now define the magnetization (per lattice site)

$$M = \langle \varphi_i \rangle \tag{2.5}$$

and the internal energy (per lattice site)

$$U = \frac{1}{2} \sum_j J_{ij} \langle \varphi_i \varphi_j \rangle \tag{2.6}$$

These are independent of i, by translation invariance. Next we define the susceptibility (per lattice site)

$$\chi = \sum_j \langle \varphi_i ; \varphi_j \rangle \tag{2.7}$$

and the specific heat (per lattice site)

$$C_H = \frac{1}{4} \sum_{j,k,l} J_{ij} J_{kl} \langle \varphi_i \varphi_j ; \varphi_k \varphi_l \rangle \tag{2.8}$$

It is *formally* true that $\chi = \partial M/\partial H$ and $C_H = J\, \partial U/\partial J$; but the rigorous proof of these "fluctuation–dissipation relations" (or "sum rules") requires a nontrivial interchange of differentiation with the infinite-volume limit. The required proof is carried out in the Appendix.

Next we define, for each $\phi > 0$, the correlation length of order ϕ,

$$\xi_\phi = \left(\chi^{-1} \sum_j |j|^\phi \langle \varphi_0 ; \varphi_j \rangle \right)^{1/\phi} \tag{2.9}$$

and the exponential ("true") correlation length

$$\xi = \limsup_{|j| \to \infty} \left(-|j| / \log \langle \varphi_0 ; \varphi_j \rangle \right) \tag{2.10}$$

By Hölder's inequality, ξ_ϕ increases with ϕ. For models obeying reflection positivity,[48] it can be shown[49] that

$$\xi \geqslant \text{const}(\phi, d)\, \xi_\phi \tag{2.11}$$

for all $\phi > 0$.

Finally, we define the higher-order cumulants

$$\bar{u}_n = \sum_{i_2, \ldots, i_n} \langle \varphi_{i_1} ; \varphi_{i_2} ; \ldots ; \varphi_{i_n} \rangle \tag{2.12}$$

with the usual definition of the truncated (connected) correlation [indicated by semicolons in (2.12)]. *Formally* $\bar{u}_n = \partial^{n-1}M/\partial H^{n-1}$; the rigorous proof requires a fluctuation–dissipation theorem. Note that the GHS inequality implies that $\bar{u}_3 \leqslant 0$.

We now make the usual assumption that there exists a "normal critical point"[4] at $J = J_c$, $H = 0$. We shall not spell out in detail here precisely what is being assumed; this will be clear from each proof. For $J > J_c$, we take $+$ boundary conditions (b.c.) in order to induce a spontaneous magnetization.

To define the critical exponents, we write as usual[2-4] $f(x) \sim x^{\lambda}$ to mean that

$$\lim_{x\downarrow 0} \log f(x)/\log x \quad \text{exists and equals } \lambda$$

We then assume that

$$J < J_c\ (T > T_c) \quad H = 0 \qquad \begin{cases} \chi \sim (J_c - J)^{-\gamma} \\ C_H \sim (J_c - J)^{-\alpha} \\ \xi \sim (J_c - J)^{-\nu} \\ \xi_\phi \sim (J_c - J)^{-\nu_\phi} \\ \bar{u}_{2m} \sim (J_c - J)^{-\gamma - 2\sum_{k=2}^{m}\Delta_{2k}} \end{cases} \tag{2.13}$$

$$J > J_c\ (T < T_c) \quad H = 0 \quad (+\ \text{b.c.}) \qquad \begin{cases} \chi \sim (J - J_c)^{-\gamma'} \\ C_H \sim (J - J_c)^{-\alpha'} \\ \dfrac{\partial M}{\partial J} \sim (J - J_c)^{\beta - 1} \\ \qquad \left[\text{hence } M \sim (J - J_c)^{\beta}\right] \\ \xi \sim (J - J_c)^{-\nu'} \\ \xi_\phi \sim (J - J_c)^{-\nu'_\phi} \\ \bar{u}_n \sim (J - J_c)^{-\gamma - \sum_{k=3}^{n}\Delta'_k} \end{cases} \tag{2.14}$$

$$J = J_c\ (T = T_c) \quad H > 0 \qquad \begin{cases} \chi \sim H^{(1/\delta) - 1} \\ \qquad (\text{hence } M \sim H^{1/\delta}) \\ C_H \sim H^{-\alpha_c/\delta} \\ \dfrac{\partial U}{\partial H} \sim H^{(\zeta + 1)/\delta - 1} \\ \xi \sim H^{-\mu} \\ \xi_\phi \sim H^{-\mu_\phi} \end{cases} \tag{2.15}$$

3. INEQUALITIES OBTAINED BY BOUNDING $\sum_{j,k}\alpha_{jk}\langle\varphi_i;\varphi_j\varphi_k\rangle$

By the GKS II and GHS inequalities, we have

$$\begin{aligned} 0 &\leqslant \sum_{j,k}\alpha_{jk}\langle\varphi_i;\varphi_j\varphi_k\rangle \\ &\leqslant \sum_{j,k}\alpha_{jk}\left[\langle\varphi_j\rangle\langle\varphi_i;\varphi_k\rangle + \langle\varphi_k\rangle\langle\varphi_i;\varphi_j\rangle\right] \\ &= 2\tilde{\alpha}M\chi \end{aligned} \tag{3.1}$$

Now by the fluctuation–dissipation relation (see Appendix),

$$\frac{\partial M}{\partial J} = \frac{1}{2}\sum_{j,k}\alpha_{jk}\langle\varphi_i;\varphi_j\varphi_k\rangle \tag{3.2}$$

Hence we have

$$0 \leqslant M^{-1}\frac{\partial M}{\partial J} \leqslant \tilde{\alpha}\chi \tag{3.3}$$

provided that $M \neq 0$. It follows immediately from (3.3) and (2.14) that

$$\gamma' \geqslant 1 \tag{3.4}$$

(assuming, of course, that $\beta < \infty$). For a slightly more general deduction, which does not require $\beta < \infty$, fix $J' > J_c$ and integrate (3.3) from $J = J_c + \epsilon$ to $J = J'$ (all at $H = 0$):

$$0 \leqslant \log M(J') - \log M(J_c + \epsilon) \leqslant \tilde{\alpha}\int_{J_c+\epsilon}^{J'} \chi(J)\,dJ \tag{3.5}$$

Since $M(J') > 0$ while $M(J_c + \epsilon)\downarrow 0$ as $\epsilon\downarrow 0$, it follows that the right side of (3.5) diverges as $\epsilon\downarrow 0$. In terms of critical exponents we conclude (3.4). [If $\gamma' = 1$, then (3.5) also implies that $\beta \leqslant \tilde{\alpha}C_-$, where $C_- = \lim_{J\downarrow J_c}(J - J_c)\chi(J)$.]

The analogous inequality $\gamma \geqslant 1$ for the region $J < J_c$ has been proven by Glimm and Jaffe[10,11] by a similar argument, studying $\partial\chi/\partial J$ and using the Lebowitz inequality in place of $\partial M/\partial J$ and the GHS inequality.[8]

The basic inequality (3.1) can also be applied on the critical isotherm $J = J_c$. Indeed, the fluctuation–dissipation relation entails

$$\frac{1}{J}\frac{\partial U}{\partial H} = \frac{1}{2}\sum_{j,k}\alpha_{jk}\langle\varphi_i;\varphi_j\varphi_k\rangle \tag{3.6}$$

so from (3.1) and (2.15) we conclude immediately that

$$\zeta \geqslant 1 \tag{3.7}$$

[8]The Glimm–Jaffe result is slightly stronger than ours: they prove that $\chi \geqslant \text{const} \times (J_c - J)^{-1}$, while we prove only that $\chi(J)$ is nonintegrable as $J\downarrow J_c$.

This bound, like (3.4), becomes an equality when the exponents take on their classical (mean-field) values.

Obtaining upper bounds on γ' and ζ is a somewhat lengthier affair. We first apply a special case of the new Lebowitz inequality:

$$\langle \varphi_i ; \varphi_j \varphi_k \rangle \geqslant |\langle \varphi_j \rangle \langle \varphi_i ; \varphi_k \rangle - \langle \varphi_k \rangle \langle \varphi_i ; \varphi_j \rangle| \tag{3.8}$$

[This inequality also follows from the Ginibre inequality $\langle q_i q_j t_k \rangle \geqslant 0$ and permutations. It improves the GKS II inequality used in (3.1), since GKS II says merely that the left side of (3.8) is nonnegative.] Setting $i = 0$ and $G(l) = \langle \varphi_0 ; \varphi_l \rangle$, we conclude that

$$\sum_{j,k} \alpha_{jk} \langle \varphi_0 ; \varphi_j \varphi_k \rangle \geqslant M \sum_{j,k} \alpha_{jk} \epsilon_{jk} [G(j) - G(k)] \tag{3.9}$$

for any choice of $\{ \epsilon_{jk} = \pm 1 \}$. To estimate (3.9) effectively, we must make a guess, for each pair j, k, as to the likely sign of $G(j) - G(k)$. A reasonable guess is that the correlations decrease with (Euclidean) distance, i.e., that $G(j) - G(k)$ has the same sign as $|k| - |j|$. This guess may not be exact (after all, the lattice theory is not Euclidean invariant[9]), but it is good enough for our purposes.[10]

For geometric simplicity, we restrict ourselves to a nearest-neighbor interaction, so that only terms with $|j - k| = 1$ occur in the sum. (This restriction will also be used in a more fundamental way later in the proof.) Taking

$$\epsilon_{jk} = \operatorname{sgn}(|k| - |j|) \tag{3.10}$$

we find that the sum in (3.9) telescopes (draw a picture!); since $G(l) \to 0$ as $|l| \to \infty$, all that remains is a sum over the hyperplanes $j_a = 0$, $1 \leqslant a \leqslant d$:

$$\sum_{j,k} \alpha_{jk} \epsilon_{jk} [G(j) - G(k)] = 4 \sum_{a=1}^{d} \sum_{j : j_a = 0} \alpha_{e_a 0} G(j) \tag{3.11}$$

$$= 2\tilde{\alpha} \sum_{j : j_1 = 0} G(j) \tag{3.12}$$

in the isotropic case. We wish to convert this sum over a hyperplane into a sum over the entire lattice.

Note first[4] that since $G(j) \geqslant 0$,

$$\sum_{|j| > R} G(j) \leqslant \sum_{|j| > R} \left(\frac{|j|}{R} \right)^{\phi} G(j) \leqslant \sum_{j} \left(\frac{|j|}{R} \right)^{\phi} G(j) = \left(\frac{\xi_\phi}{R} \right)^{\phi} \chi \tag{3.13}$$

[9] Although it is likely to become so in the limit of the critical point.[50,51]

[10] In the case of a nearest-neighbor interaction, so that only $|j - k| = 1$ occurs in the sum, the guess *is* exact, according to a theorem of Schrader,[38] Messager and Miracle-Sole,[39] and Hegerfeldt.[40]

Hence

$$\sum_{|j| \leqslant R} G(j) \geqslant \left[1 - \left(\frac{\xi_\phi}{R}\right)^\phi\right] \chi \tag{3.14}$$

Next, an inequality due to Schrader,[(38)] Messager and Miracle-Sole,[(39)] and Hegerfeldt[(40)] tells us that

$$G(0, j_2, \ldots, j_d) \geqslant G(j_1, j_2, \ldots, j_d) \tag{3.15}$$

(Here we really use the assumption of a nearest-neighbor interaction.) Hence

$$\sum_{j_1=0} G(j) \geqslant \frac{1}{2R+1} \sum_{|j_1| \leqslant R} G(j) \geqslant \frac{1}{2R+1} \sum_{|j| \leqslant R} G(j)$$

$$\geqslant \frac{1}{2R+1}\left[1 - \left(\frac{\xi_\phi}{R}\right)^\phi\right] \chi \tag{3.16}$$

Taking $R = 2\xi_\phi$, we conclude from (3.16), (3.11), and (3.9) that

$$\sum_{j,k} \alpha_{jk} \langle \varphi_0; \varphi_j \varphi_k \rangle \geqslant \text{const}(\phi) \tilde{\alpha} M \chi / (1 + \xi_\phi) \tag{3.17}$$

for any $\phi > 0$. This, together with (3.2), (3.6), (2.14), and (2.15) immediately yields the critical-exponent inequalities[11]

$$\gamma' \leqslant 1 + \nu'_\phi \tag{3.18}$$

and

$$\zeta \leqslant 1 + \delta \mu_\phi \tag{3.19}$$

The foregoing argument can undoubtedly be generalized to handle a variety of non-nearest-neighbor interactions. For example, if the interaction satisfies reflection positivity, then (3.15) is valid at least when summed over $j_2, \ldots, j_d$; this is all we really use in (3.16)!

It is worth noting that the inequalities (3.18) and (3.19) are rather poor [as is to be expected from the rather unphysical use of the inequalities (3.8) and (3.9)]. For example, in the $d = 2$ Ising model, $\gamma' = 7/4$ while $\nu'_\phi = 1$; and in the $d = 3$ Ising model, $\gamma' = 1.25 \pm 0.01$ while $\nu'_\phi = 0.638 \pm 0.008$.[(17,18,52–54)12] Moreover, in mean-field theory (believed to be accurate for $d > 4$), $\gamma' = 1$ while $\nu'_\phi = 1/2$.

[11]This assumes, of course that $\nu'_\phi \geqslant 0$ and $\mu_\phi \geqslant 0$. Those readers unwilling to take such plausible assumptions on faith can consult Sections 4 and 5 for proofs.

[12]Actually, I have cheated a little here: the quoted values are *inferred* from the presumed scaling relations $\gamma' = \gamma$ and $\nu'_\phi = \nu_2$ together with high-temperature series computations of γ and ν_2.

4. INEQUALITIES OBTAINED BY BOUNDING $\sum_{j,k}\langle\varphi_i;\varphi_j;\varphi_k\rangle$

It follows immediately from the GHS inequality that

$$\sum_{j,k}\langle\varphi_i;\varphi_j;\varphi_k\rangle \leqslant 0 \tag{4.1}$$

Now the fluctuation–dissipation relation tells us that this quantity equals $\partial\chi/\partial H(=\partial^2 M/\partial H^2)$; thus χ increases as $H\downarrow 0$ at fixed J (in particular, at $J=J_c$). Hence, by (2.15),

$$\delta \geqslant 1 \tag{4.2}$$

To get a further bound, we rewrite the new Lebowitz (or Ginibre) inequality (3.8) in the form

$$\langle\varphi_i;\varphi_j;\varphi_k\rangle \geqslant -2\min\left[\langle\varphi_j\rangle\langle\varphi_i;\varphi_k\rangle, \langle\varphi_k\rangle\langle\varphi_i;\varphi_j\rangle\right] \tag{4.3}$$

We set $i=0$, and again make the crude guess that correlations decrease with (Euclidean) distance. That is, we deduce from (4.3) that

$$-\sum_{j,k}\langle\varphi_0;\varphi_j;\varphi_k\rangle \leqslant 4M\sum_j \sum_{\substack{k\\ |k|\leqslant|j|}} \langle\varphi_0;\varphi_j\rangle \tag{4.4}$$

Since the number of lattice points k with $|k|\leqslant|j|$ is bounded by $c_1+c_2|j|^d$, we conclude that

$$-\sum_{j,k}\langle\varphi_0;\varphi_j;\varphi_k\rangle \leqslant 4M\left(c_1\chi + c_2\xi_d^d\chi\right) \tag{4.5}$$

or

$$-\chi^{-1}\frac{\partial\chi}{\partial H} \leqslant M\left(c_3 + c_4\xi_d^d\right) \tag{4.6}$$

provided that $\chi\neq 0$. Now fix $J=J_c$ and $H'>0$, and integrate (4.6) from $H=\epsilon$ to $H=H'$:

$$\log\chi(\epsilon) - \log\chi(H') \leqslant \int_\epsilon^{H'} M(H)\left[c_3 + c_4\xi_d(H)^d\right]dH \tag{4.7}$$

Since $\chi(H')<\infty$ while $\chi(\epsilon)\uparrow\infty$ as $\epsilon\downarrow 0$ (provided that $\delta>1$), it follows that the right side of (4.7) diverges as $\epsilon\downarrow 0$. In terms of critical exponents, we conclude that

$$d\mu_d \geqslant 1 + 1/\delta \tag{4.8}$$

Since ξ_ϕ (and hence μ_ϕ) increases with ϕ, it follows that

$$d\mu_\phi \geqslant 1 + 1/\delta \qquad \text{for } \phi \geqslant d \tag{4.9}$$

If, in addition, reflection positivity holds, so that $\mu \geqslant \mu_\phi$ for all ϕ, we then have also

$$d\mu \geqslant 1 + 1/\delta \tag{4.10}$$

Inequality (4.9) is actually not new: it has been proven by Liu and Stanley[(55)] for *all* $\phi \geqslant 0$.[13] But their proof uses the GHS inequality, while the foregoing proof uses only the more widely applicable new Lebowitz (or Ginibre) inequality.

Inequality (4.5) can also be applied at $J > J_c$, $H = 0$; the immediate conclusion is that[14]

$$\Delta_3' \leqslant d\nu_\phi' - \beta \qquad \text{for } \phi \geqslant d \tag{4.11}$$

Indeed, (4.5) is an absolute upper bound on the "dimensionless renormalized three-point coupling constant"

$$g_\phi^{(3)} = -\bar{u}_3 / M\chi\xi_\phi^d \tag{4.12}$$

This bound is analogous to the bounds of Glimm–Jaffe[(56)] and others[(22,57)] for the four-point coupling constant.

The critical-exponent inequalities (4.8)–(4.11) become equalities if hyperscaling holds.

5. INEQUALITIES OBTAINED BY BOUNDING $\sum_{j,k,l}\alpha_{kl}\langle\varphi_i;\varphi_j;\varphi_k\varphi_l\rangle$

Let us consider the quantity

$$\frac{\partial\chi}{\partial J} = \frac{1}{2}\sum_{j,k,l}\alpha_{kl}\langle\varphi_i;\varphi_j;\varphi_k\varphi_l\rangle \tag{5.1}$$

By the new Lebowitz inequality,

$$\langle\varphi_i;\varphi_j;\varphi_k\varphi_l\rangle \geqslant -2\min\left[\langle\varphi_i\rangle\langle\varphi_j;\varphi_k\varphi_l\rangle, \langle\varphi_j\rangle\langle\varphi_i;\varphi_k\varphi_l\rangle\right] \tag{5.2}$$

Now, by translation invariance, we can choose to fix k instead of i in (5.1), and sum over the rest; we do so, setting $k = 0$. We choose the first term in the brackets of (5.2) if $|j| \geqslant |i|$, and the second term otherwise. Then

$$-\frac{\partial\chi}{\partial J} \leqslant 2M\sum_{j,l}\sum_{\substack{i \\ |i|\leqslant|j|}}\alpha_{0l}\langle\varphi_j;\varphi_0\varphi_l\rangle \tag{5.3}$$

$$\leqslant 2M\sum_{j,l}\alpha_{0l}\left(c_1 + c_2|j|^d\right)\langle\varphi_j;\varphi_0\varphi_l\rangle \tag{5.4}$$

$$= 4c_1 M\frac{\partial M}{\partial J} + 2c_2 M\sum_{j,l}\alpha_{0l}|j|^d\langle\varphi_j;\varphi_0\varphi_l\rangle \tag{5.5}$$

Now by the GKS II and GHS inequalities,

$$0 \leqslant \langle\varphi_j;\varphi_0\varphi_l\rangle \leqslant M\left(\langle\varphi_j;\varphi_0\rangle + \langle\varphi_j;\varphi_l\rangle\right) \tag{5.6}$$

[13]They actually claim it to be true for all real ϕ, but I am unable to make any sense of $\phi \leqslant 0$.

[14]This assumes, of course that $\nu_\phi' \geqslant 0$ for $\phi \geqslant d$. Those readers unwilling to take such a plausible assumption on faith can consult Section 5 for a proof.

Hence

$$\langle \varphi_j ; \varphi_0 \varphi_l \rangle \leqslant \langle \varphi_j ; \varphi_0 \varphi_l \rangle^{1-\lambda} M^{\lambda} (\langle \varphi_j ; \varphi_0 \rangle + \langle \varphi_j ; \varphi_l \rangle)^{\lambda} \tag{5.7}$$

for $0 < \lambda \leqslant 1$. Using (5.7) and Hölder's inequality [with $p = 1/(1-\lambda)$ and $q = 1/\lambda$], we obtain

$$\sum_{j,l} \alpha_{0l} |j|^d \langle \varphi_j ; \varphi_0 \varphi_l \rangle \leqslant M^{\lambda} \Big[\sum_{j,l} \alpha_{0l} \langle \varphi_j ; \varphi_0 \varphi_l \rangle \Big]^{1-\lambda}$$

$$\times \Big[\sum_{j,l} \alpha_{0l} |j|^{d/\lambda} (\langle \varphi_j ; \varphi_0 \rangle + \langle \varphi_j ; \varphi_l \rangle) \Big]^{\lambda} \tag{5.8}$$

$$\leqslant M^{\lambda} \Big(\frac{\partial M}{\partial J} \Big)^{1-\lambda} \Big[\sum_l \alpha_{0l} \big(c_3 \chi \xi_{d/\lambda}^{d/\lambda} + c_4 |l|^{d/\lambda} \chi \big) \Big]^{\lambda} \tag{5.9}$$

$$\leqslant M^{\lambda} \Big(\frac{\partial M}{\partial J} \Big)^{1-\lambda} \chi^{\lambda} \big[c_5 + c_6 \xi_{\phi}^d \big] \tag{5.10}$$

for $\phi \geqslant d/\lambda$, where we have used $|j|^{d/\lambda} \leqslant \text{const}(|j-l|^{d/\lambda} + |l|^{d/\lambda})$ in going from (5.8) to (5.9), and have assumed that $\sum_l |l|^{d/\lambda} \alpha_{0l} < \infty$ in going from (5.9) to (5.10). Inserting (5.10) into (5.5), we conclude that

$$-\frac{\partial \chi}{\partial J} \leqslant c_7 M \frac{\partial M}{\partial J} + c_8 M^{1+\lambda} \Big(\frac{\partial M}{\partial J} \Big)^{1-\lambda} \chi^{\lambda} \big[1 + \xi_{\phi}^d \big] \tag{5.11}$$

for $\phi \geqslant d/\lambda$ and $0 < \lambda \leqslant 1$.

We now apply (5.11) to deduce an inequality on the critical exponents for $J \downarrow J_c$ at $H = 0$. Note first that the inequality $\gamma' \geqslant 1$ (see Section 3) implies that

$$\chi \geqslant \text{const} \times (J - J_c)^{-(1-\epsilon)} \qquad \text{for } J - J_c \text{ small} \tag{5.12}$$

for any $\epsilon > 0$; hence the second term on the right side of (5.11) dominates over the first (up to a possible power ϵ):

$$-\frac{\partial \chi}{\partial J} \leqslant \text{const} \times M^{1+\lambda} \Big(\frac{\partial M}{\partial J} \Big)^{1-\lambda} \chi^{\lambda} \big[1 + \xi_{\phi}^d \big] (J - J_c)^{-\epsilon} \tag{5.13}$$

for $J - J_c$ small. Multiplying both sides by $\chi^{-\lambda}$ and integrating from $J = J'$ down to $J = J_c + \epsilon'$, we get

$$\chi(J_c + \epsilon')^{1-\lambda} - \chi(J')^{1-\lambda} \leqslant \text{const} \times \int_{J_c + \epsilon'}^{J'} M^{1+\lambda} \Big(\frac{\partial M}{\partial J} \Big)^{1-\lambda} \big[1 + \xi_{\phi}^d \big]$$

$$\times (J - J_c)^{-\epsilon} \, dJ \tag{5.14}$$

(with $\log \chi$ replacing $\chi^{1-\lambda}$ if $\lambda = 1$). Since $\chi(J') < \infty$ while $\chi(J_c + \epsilon') \sim (\epsilon')^{-\gamma'}$ as $\epsilon' \downarrow 0$, it follows that the right side of (5.14) diverges at least as rapidly as $(\epsilon')^{-(1-\lambda)\gamma'}$ (or as $|\log \epsilon'|$ if $\lambda = 1$) as $\epsilon' \downarrow 0$. Hence, in terms of critical exponents,

$$\max[d\nu'_\phi, 0] + \epsilon + 1 - \lambda - 2\beta \geqslant (1-\lambda)\gamma' + 1 \tag{5.15}$$

for $\phi \geqslant d/\lambda$ and all $\epsilon > 0$. Since $\gamma' \geqslant 1$, the optimal value of λ is the minimal permitted one, $\lambda = d/\phi$. Hence

$$\max[d\nu'_\phi, 0] \geqslant \gamma' + 2\beta - \frac{d}{\phi}(\gamma' - 1) \qquad \text{for } \phi \geqslant d \tag{5.16}$$

Since the right side of (5.16) surely exceeds zero, we conclude that

$$d\nu'_\phi \geqslant \gamma + 2\beta - \frac{d}{\phi}(\gamma' - 1) \qquad \text{for } \phi \geqslant d \tag{5.17}$$

Finally, Rushbrooke's[1] inequality $\gamma' + 2\beta \geqslant 2 - \alpha'$ implies that

$$d\nu'_\phi \geqslant 2 - \alpha' - \frac{d}{\phi}(\gamma' - 1) \qquad \text{for } \phi \geqslant d \tag{5.18}$$

a weak form of the Josephson[5] inequality.

The "correction term" $-(d/\phi)(\gamma' - 1)$ in (5.17) and (5.18) is disconcerting; I would like to eliminate it, but I do not know how. Of course, if $\gamma' < \infty$, we can take $\phi \to \infty$ and conclude that

$$d\nu'_\infty \geqslant \gamma' + 2\beta \geqslant 2 - \alpha' \tag{5.19}$$

where

$$\nu'_\infty \equiv \lim_{\phi \to \infty} \nu'_\phi = \sup_\phi \nu'_\phi \tag{5.20}$$

Moreover, if reflection positivity holds, (2.11) implies that $\nu' \geqslant \nu'_\infty$; hence

$$d\nu' \geqslant \gamma' + 2\beta \geqslant 2 - \alpha' \tag{5.21}$$

The inequality $d\nu' \geqslant \gamma' + 2\beta$ has been proposed previously by Choy and Ree,[58] who gave a nonrigorous argument for it.

The *equality* $d\nu' = \gamma' + 2\beta$ is a hyperscaling relation. One useful interpretation is the following: Imagine trying to construct a scaling limit as $J \downarrow J_c$.[50] Lengths must be shrunk by a factor $\xi \sim (J - J_c)^{-\nu'}$; and the spin must be rescaled by a factor $M^{-1} \sim (J - J_c)^{-\beta}$ to preserve the normalization of the one-point correlation function. But this means that the spatial integral of the two-point (connected) correlation function (i.e., the susceptibility) is rescaled by a factor $M^{-2}\xi^{-d} \sim (J - J_c)^{d\nu' - 2\beta}$. Hence the connected two-point function has a finite, nonzero limit only if $d\nu' - 2\beta - \gamma' = 0$. If $d\nu' - 2\beta - \gamma' > 0$, then the connected two-point function *vanishes* in the scaling limit. [A similar interpretation can be given for inequalities (4.8)–(4.11).]

The arguments of the present section can also be applied on the critical isotherm $J = J_c$ to deduce the critical-exponent inequality

$$d\mu_\phi \geqslant 1 + \frac{1}{\delta} - \frac{d}{\phi}\,\frac{\zeta - 1}{\delta} \qquad \text{for } \phi \geqslant d \tag{5.22}$$

However, this is weaker than the inequality (4.9) already proven by a different method.

6. SOME MISCELLANEOUS INEQUALITIES FOR GAP EXPONENTS

In this section, we shall prove some miscellaneous inequalities for the gap exponents Δ_{2m} $(m \geqslant 2)$. The first follows from a rather strange inequality due to GHS,[42] valid for the spin-1/2 Ising model with ferromagnetic pair interactions and zero magnetic field[15]:

$$\begin{aligned}\langle\varphi_i\varphi_j\varphi_k\varphi_l\rangle - \langle\varphi_i\varphi_j\rangle\langle\varphi_k\varphi_l\rangle - \langle\varphi_i\varphi_k\rangle\langle\varphi_j\varphi_l\rangle - \langle\varphi_i\varphi_l\rangle\langle\varphi_j\varphi_k\rangle \\ \leqslant -2\langle\varphi_i\varphi_j\rangle\langle\varphi_i\varphi_k\rangle\langle\varphi_i\varphi_l\rangle\end{aligned} \tag{6.1}$$

For $J < J_c$ (so that $M = 0$), the left side of (6.1) is just $\langle\varphi_i;\varphi_j;\varphi_k;\varphi_l\rangle$. Hence, summing over j, k, l and using (2.12), we conclude that

$$\bar{u}_4 \leqslant -2\chi^3 \tag{6.2}$$

(This improves the Lebowitz inequality $\bar{u}_4 \leqslant 0$.) Hence, in terms of the critical exponents (2.13),

$$\Delta_4 \geqslant \gamma \tag{6.3}$$

Actually, (6.3) is rather poor: for example, in mean-field theory $\Delta_4 = 3/2$ while $\gamma = 1$.

A somewhat more interesting family of inequalities for gap exponents can be deduced from a set of correlation inequalities due to Newman,[41] derived from the Lee–Yang theorem. Consider a model with ferromagnetic pair interactions and zero magnetic field (and zero boundary condition); and assume that each single-spin measure ν_i is even and satisfies the Lee–Yang condition:

$$\text{If } \operatorname{Im} z \neq 0, \quad \text{then } \int \exp(z\varphi)\,d\nu_i(\varphi) \neq 0 \tag{6.4}$$

Then Newman shows that any random variable $X = \sum_i \lambda_i \varphi_i$ with all $\lambda_i \geqslant 0$ has a representation of the following form:

$$\langle\exp(zX)\rangle = \exp(bz^2)\prod_j\left(1 + z^2/\alpha_j^2\right) \tag{6.5}$$

for some $b \geqslant 0$ and $0 < \alpha_1 \leqslant \alpha_2 \leqslant \cdots$, with $\sum_j \alpha_j^{-2} < \infty$; here the set $\{\alpha_j\}$ may be empty, finite, or infinite. It follows from (6.5) that the $2m$-fold cumulant

$$u_{2m}(X) = \underbrace{\langle X; \ldots ; X\rangle}_{2m \text{ times}} = \frac{\partial^{2m}}{\partial z^{2m}} \log\langle\exp(zX)\rangle\Big|_{z=0} \tag{6.6}$$

[15] Note that this inequality is *not* multilinear in the spins φ_i. Hence it does *not* extend to models other than spin − 1/2.

is given by

$$u_{2m}(X) = 2b\delta_{m1} + (-1)^{m-1}\frac{(2m)!}{m}\sum_j \alpha_j^{-2m} \tag{6.7}$$

for m $\geqslant$ 1. Defining for simplicity

$$v_{2m}(X) = (-1)^{m-1}\frac{m}{(2m)!}u_{2m}(X) \tag{6.8}$$

Hölder's inequality applied to (6.7) immediately gives

$$v_{2m}(X) \leqslant \prod_{k=1}^{l} v_{2m_k}(X)^{\beta_k} \tag{6.9}$$

where $m = \beta_1 m_1 + \cdots + \beta_l m_l$, $\beta_k \geqslant 0$ for all k, and $\beta_1 + \cdots + \beta_l \geqslant 1$.

To apply (6.9), consider a model in finite volume V with periodic boundary conditions (so as to ensure translation invariance), and let $X_V = \sum_i \varphi_i$. Then as $V \to \infty$, $u_{2m}(X_V)/V$ approaches what we have called $\bar{u}_{2m}$ [see (2.12)]. Hence we get a nontrivial inequality from (6.9) as $V \to \infty$ only if $\beta_1 + \cdots + \beta_l = 1$. All such inequalities are deducible from the subset

$$|\bar{u}_{2m}| \leqslant \text{const}(m)\left[\bar{u}_{2m-2}\bar{u}_{2m+2}\right]^{1/2} \tag{6.10}$$

with $m \geqslant 2$. Inserting the critical-exponent definition (2.13), we conclude that

$$\Delta_{2m} \leqslant \Delta_{2m+2} \qquad \text{for } m \geqslant 2 \tag{6.11}$$

i.e., $\Delta_4 \leqslant \Delta_6 \leqslant \Delta_8 \leqslant \cdots$. (Of course, if the scaling relations hold, then the gap exponents Δ_{2m} are all *equal*.)

The inequality (6.11) was first proven by Baker[(59)] for the spin-1/2 Ising ferromagnet, by essentially the same method as above. However, the proof based on Newman's inequalities has the advantage that it generalizes immediately to any model satisfying the Lee–Yang condition (6.4), for example, the φ^4 lattice field theory.[(37)] It would be interesting to try to extend this method to positive magnetic field (or positive boundary conditions), so as to treat $J > J_c$ and $J = J_c$; but I have not succeeded in doing so.[16]

While on the subject of gap–exponent inequalities, it is worth remarking that Lieb and the present author have proven[(57)] the inequality

$$d\nu_\phi \geqslant 2\Delta_4 - \gamma \qquad \text{for } \phi \geqslant d/2 \tag{6.12}$$

[16]Note that with the usual definitions[(2)] $\Delta_1' = 2 - \alpha' - \beta$ and $\Delta_2' = \beta + \gamma'$, the inequality $\Delta_1' \leqslant \Delta_2'$ is just Rushbrooke's inequality[(1)] $\alpha' + 2\beta + \gamma' \geqslant 2$. If the inequality $\Delta_2' \leqslant \Delta_3'$ could be proven, then the inequality $d\nu_\phi' \geqslant \gamma' + 2\beta$ (for $\phi \geqslant d$), which improves (5.17), would follow immediately from $\Delta_3' \leqslant d\nu_\phi' - \beta$, which is (4.11).

as a by-product of our study of rigorous numerical upper bounds for the "dimensionless renormalized 4-point coupling constant" $g_\phi^{(4)} = -\bar{u}_4/\chi^2\xi_\phi^d$. This strengthens earlier results of Schrader[22] and Glimm and Jaffe.[56] Also, Glimm and Jaffe[56] have shown that

$$d\nu \geqslant \Delta_4 + \Delta_6 - \gamma \tag{6.13}$$

(at least in a Euclidean-invariant field theory). It would be desirable to extend (6.13) to lattice models, using a suitable ν_ϕ in place of ν. It would also be desirable to prove the stronger inequality $d\nu \geqslant 2\Delta_6 - \gamma$ (or $d\nu_\phi \geqslant 2\Delta_6 - \gamma$ for suitable ϕ); I suspect that this can be done by using the strong form of the Gaussian inequality[45,46] together with the inequality $\mu_6 \geqslant 0$,[46] but I have not worked out the details.

APPENDIX: PROOF OF A GENERAL FLUCTUATION–DISSIPATION RELATION

The purpose of this Appendix is to give a rigorous proof of the "fluctuation–dissipation relations" (or "sum rules") which are used in the present paper. The ideas in this Appendix are due largely to Jean Bricmont, who has kindly given me permission to include them here. For previous work of a similar nature, see Refs. 4, 27, and 60–62.

Let us fix once and for all the boundary condition b; having chosen it, we shall drop it from the notation. Let us also fix an increasing sequence $\{\Lambda_n\}_{n\geqslant 1}$ of finite subsets of $\mathfrak{L}$ with union equal to $\mathfrak{L}$; and fix the single-spin measures $\{\nu_i\}$. Now let $\{\Phi^{(\lambda)}\}_{0\leqslant\lambda\leqslant 1}$ be a family of interactions, with

$$\Phi_X^{(\lambda)} = \Phi_X^{(0)} + \lambda\Phi'_X \tag{A.1}$$

for each X. Then, for each Λ, let f_Λ be a function of the spins $\{\varphi_i\}_{i\in\mathfrak{L}}$, and assume that f_Λ and Φ' are sufficiently well behaved that the fluctuation–dissipation relation holds in finite volume, i.e.,

$$\frac{d}{d\lambda}\langle f_\Lambda\rangle_\Lambda^{\Phi^{(\lambda)}} = -\sum_{X\cap\Lambda\neq\varnothing}\langle f_\Lambda;\Phi'_X\rangle_\Lambda^{\Phi^{(\lambda)}} \tag{A.2}$$

with the right side continuous in λ. [This is just an extremely mild integrability condition. (A.2) follows essentially trivially from (2.1).]

Assume further that

$$F(\lambda) \equiv \lim_{n\to\infty}\langle f_{\Lambda_n}\rangle_{\Lambda_n}^{\Phi^{(\lambda)}} \tag{A.3}$$

and

$$F_X(\lambda) \equiv \lim_{n\to\infty}\langle f_{\Lambda_n};\Phi'_X\rangle_{\Lambda_n}^{\Phi^{(\lambda)}} \tag{A.4}$$

exist for all X and $0 \leqslant \lambda \leqslant 1$. [Often there will be a function f_∞ such that

$$F(\lambda) = \langle f_\infty\rangle_\infty^{\Phi^{(\lambda)}} \tag{A.5}$$

and

$$F_X(\lambda) = \langle f_\infty ; \Phi'_X \rangle_\infty^{\Phi(\lambda)} \tag{A.6}$$

In fact, we shall usually take $f_\Lambda = f$ for all Λ, where f is the quantity to be differentiated. Then (A.5) and (A.6) hold with $f_\infty = f$, simply by definition of $\langle \cdots \rangle_\infty^{\Phi(\lambda)}$ (at least if f is reasonable).]

We then have the following general fluctuation–dissipation theorem:

Proposition A.1. Assume that

$$\lim_{\lambda \downarrow 0} F_X(\lambda) = F_X(0) \tag{A.7}$$

for each X; and assume further that

$$\sum_X \sup_{0 \leqslant \lambda \leqslant \lambda_0} \sup_{n \geqslant n_0} |\langle f_{\Lambda_n} ; \Phi'_X \rangle_{\Lambda_n}^{\Phi(\lambda)}| < \infty \tag{A.8}$$

for some n_0 and some $\lambda_0 > 0$. Then

$$\frac{d}{d\lambda} F(\lambda)\bigg|_{\lambda=0+} = -\sum_X F_X(0) \tag{A.9}$$

Proof. By definition,

$$\begin{aligned}\frac{d}{d\lambda} F(\lambda)\bigg|_{\lambda=0+} &= \lim_{\lambda \downarrow 0} \lambda^{-1}[F(\lambda) - F(0)] \\ &= \lim_{\lambda \downarrow 0} \lim_{n \to \infty} \lambda^{-1}\left[\langle f_{\Lambda_n} \rangle_{\Lambda_n}^{\Phi(\lambda)} - \langle f_{\Lambda_n} \rangle_{\Lambda_n}^{\Phi(0)}\right]\end{aligned}$$

Then by (A.2),

$$\begin{aligned}\frac{d}{d\lambda} F(\lambda)\bigg|_{\lambda=0+} &= \lim_{\lambda \downarrow 0} \lim_{n \to \infty} \lambda^{-1} \int_0^\lambda d\lambda' \frac{d}{d\lambda'} \langle f_{\Lambda_n} \rangle_{\Lambda_n}^{\Phi(\lambda')} \\ &= -\lim_{\lambda \downarrow 0} \lim_{n \to \infty} \lambda^{-1} \int_0^\lambda d\lambda' \sum_{X \cap \Lambda_n \neq \varnothing} \langle f_{\Lambda_n} ; \Phi'_X \rangle_{\Lambda_n}^{\Phi(\lambda')}\end{aligned}$$

Now, by (A.8) and the dominated convergence theorem, we can (for $\lambda \leqslant \lambda_0$) take $n \to \infty$ inside the integral over λ' and the sum over X:

$$\frac{d}{d\lambda} F(\lambda)\bigg|_{\lambda=0+} = -\lim_{\lambda \downarrow 0} \lambda^{-1} \int_0^\lambda d\lambda' \sum_X F_X(\lambda')$$

Moreover, (A.8) also implies that

$$\sum_X \sup_{0 \leqslant \lambda' \leqslant \lambda_0} |F_X(\lambda')| < \infty$$

Hence, for $\lambda \leqslant \lambda_0$, we can interchange the integral over λ' with the sum over X; moreover, we can then take $\lambda\downarrow 0$ inside the summation:

$$\frac{d}{d\lambda}F(\lambda)\bigg|_{\lambda=0+} = -\sum_X \lim_{\lambda\downarrow 0}\lambda^{-1}\int_0^\lambda d\lambda' F_X(\lambda')$$

But then (A.9) follows immediately from (A.7). ■

We are thus reduced to verifying, in any given case, the hypotheses (A.7) and (A.8). Hypothesis (A.7) says essentially that the measures $\langle\cdots\rangle_\infty^{\Phi^{(\lambda)}}$ converge (on suitable observables) to $\langle\cdots\rangle_\infty^{\Phi^{(0)}}$ as $\lambda\downarrow 0$ [if (A.6) holds, it says exactly this]. Now this is exceedingly natural, but it is *not* trivial: For example, let $\Phi^{(\lambda)}$ be the interaction for the Ising model somewhere below the critical temperature, in magnetic field λ, and use *minus* boundary conditions (i.e., $b_i = -1$ for all i). Then $\lim_{\lambda\downarrow 0}\langle\cdots\rangle_\infty^{\Phi^{(\lambda)}}$ is the zero-field state with positive magnetization; it does *not* equal $\langle\cdots\rangle_\infty^{\Phi^{(0)}} \equiv \lim_{n\to\infty}\langle\cdots\rangle_{\Lambda_n,b}^{\Phi^{(0)}}$, since this is the zero-field state with *negative* magnetization.

We therefore have to verify (A.7) on a case-by-case basis.[17] In any event, (A.7) is not an unreasonable requirement: if expectation values are not even continuous as $\lambda\downarrow 0$, how can we expect them to be differentiable?

It remains, therefore, to find methods for demonstrating the uniform summability (A.8). In general, we expect (A.8) to hold (for reasonable perturbations Φ') whenever $\Phi^{(0)}$ is not a critical point. We illustrate this in a number of cases; our method is to use correlation inequalities to reduce (A.8) to a quantity involving only the two-point (connected) correlation function.

For purposes of illustration, we assume that f_Λ is a finite product of spins φ^A, and that Φ' is generated [according to (2.2)] by a single term $-J_B\varphi^B$ and its translates, i.e.,

$$\Phi_X = \begin{cases} -J_B\varphi^{B+i} & \text{if } X = \operatorname{supp}(B+i) \\ 0 & \text{otherwise} \end{cases} \tag{A.10}$$

where $B+i$ is the translate of B by $i \in \mathfrak{L}$. Then (A.8) becomes the claim that

$$\sum_{i\in\mathfrak{L}} \sup_{0\leqslant\lambda\leqslant\lambda_0} \sup_{n\geqslant n_0} |\langle\varphi^A;\varphi^{B+i}\rangle_{\Lambda_n}^{\Phi^{(\lambda)}}| < \infty \tag{A.11}$$

Under a variety of conditions, we can bound $\langle\varphi^A;\varphi^{B+i}\rangle$ by two-point functions:

(1) *GHS inequality* ($|A| = 1$, $|B| = 2$ or $|A| = 2$, $|B| = 1$):

$$0 \leqslant \langle\varphi_j;\varphi_k\varphi_l\rangle \leqslant \langle\varphi_k\rangle\langle\varphi_j;\varphi_l\rangle + \langle\varphi_l\rangle\langle\varphi_j;\varphi_k\rangle. \tag{A.12}$$

[17] For example, it is easy to show[25] that if the interactions $\Phi^{(\lambda)}$ *decrease* to $\Phi^{(0)}$ in a suitable (FKG) sense (as in the example above), and if *plus* boundary conditions are used, then the convergence does hold.

Of course, we use also the GKS II inequality for the lower bound in (A.12).

(2) *Lebowitz inequality* ($|A| = |B| = 2$):

$$\begin{aligned} 0 \leqslant \langle \varphi_j \varphi_k ; \varphi_l \varphi_m \rangle &\leqslant \langle \varphi_j ; \varphi_l \rangle \langle \varphi_k ; \varphi_m \rangle + \langle \varphi_j ; \varphi_m \rangle \langle \varphi_k ; \varphi_l \rangle \\ &+ \langle \varphi_j \rangle \langle \varphi_l \rangle \langle \varphi_k ; \varphi_m \rangle + \langle \varphi_k \rangle \langle \varphi_m \rangle \langle \varphi_j ; \varphi_l \rangle \\ &+ \langle \varphi_j \rangle \langle \varphi_m \rangle \langle \varphi_k ; \varphi_l \rangle + \langle \varphi_k \rangle \langle \varphi_l \rangle \langle \varphi_j ; \varphi_m \rangle \end{aligned} \tag{A.13}$$

This method was previously used by the author.[12]

(3) *Lebowitz–Gaussian inequality* ($H = 0$, $b = 0$ only):

If $|A|$ and $|B|$ are both odd, then $\langle \varphi^A ; \varphi^{B+i} \rangle = \langle \varphi^A \varphi^{B+i} \rangle$ can be expanded by the Gaussian[45–47] or Lebowitz inequality into a sum of products of two-point functions; since $|A|$ and $|B|$ are odd, at least one of these two-point functions in each term pairs an element of A with one of $B + i$, thus exhibiting the decay as $|i| \to \infty$.

If $|A|$ and $|B|$ are both even, the aforesaid inequalities imply[47,63,64] that

$$\langle \varphi^A ; \varphi^{B+i} \rangle \leqslant \sum_{\substack{A_1 \cup A_2 = A \\ B_1 \cup B_2 = B \\ |A_1|, |B_1| \text{ odd}}} \langle \varphi^{A_1} \varphi^{B_1+i} \rangle \langle \varphi^{A_2} \varphi^{B_2+i} \rangle \tag{A.14}$$

Expanding further into two-point functions, we reach the same conclusion as above.

Finally, if $|A|$ and $|B|$ have opposite parity, then $\langle \varphi^A ; \varphi^{B+i} \rangle = 0$.

It goes without saying that $\langle \varphi^A ; \varphi^{B+i} \rangle \geqslant 0$ by GKS II.

(4) *FKG inequality*: If the spins are *bounded* in absolute value by K, then it can be shown[29,61,65] using the FKG inequality that

$$|\langle \varphi^A ; \varphi^{B+i} \rangle| \leqslant K^{|A|+|B|-2} \sum_{\substack{j \in \operatorname{supp} A \\ k \in \operatorname{supp} B}} \langle \varphi_j ; \varphi_{k+i} \rangle \tag{A.15}$$

In the case of unbounded spins, we use an argument due to Bricmont *et al.*[65]: Let $K > 0$ and define the cutoff spins

$$\sigma_j^{(K)} = (\operatorname{sgn} \varphi_j) \min(K, |\varphi_j|) \tag{A.16}$$

Then by the FKG inequality we find that

$$\begin{aligned} |\langle \sigma^{(K)A} ; \sigma^{(K)B+i} \rangle| &\leqslant K^{|A|+|B|-2} \sum_{\substack{j \in \operatorname{supp} A \\ k \in \operatorname{supp} B}} \langle \sigma_j^{(K)} ; \sigma_{k+i}^{(K)} \rangle \\ &\leqslant K^{|A|+|B|-2} \sum_{\substack{j \in \operatorname{supp} A \\ k \in \operatorname{supp} B}} \langle \varphi_j ; \varphi_{k+i} \rangle \end{aligned} \tag{A.17}$$

Now by the superstability estimate[23,66]

$$\operatorname{Prob}(|\varphi_j| \geqslant K) \leqslant c \exp(-\alpha K^2) \tag{A.18}$$

for suitable $c, \alpha > 0$. Hence

$$|\langle \varphi^A; \varphi^{B+i} \rangle - \langle \sigma^{(K)A}; \sigma^{(K)B+i} \rangle| \leqslant c' K^\beta \exp(-\alpha K^2) \tag{A.19}$$

for suitable c', β (depending on $|A|$ and $|B|$). We can now combine (A.17) and (A.19), and optimize over K; choosing

$$K = \left(-\frac{1}{\alpha} \log \sum_{\substack{j \in \operatorname{supp} A \\ k \in \operatorname{supp} B}} \langle \varphi_j; \varphi_{k+i} \rangle \right)^{1/2} \tag{A.20}$$

we find that

$$|\langle \varphi^A; \varphi^{B+i} \rangle| \leqslant \text{const}\, F\left(\sum_{\substack{j \in \operatorname{supp} A \\ k \in \operatorname{supp} B}} \langle \varphi_j; \varphi_{k+i} \rangle \right) \tag{A.21}$$

with

$$F(x) = x |\log x|^\gamma \tag{A.22}$$

for suitable power γ (depending on $|A|$ and $|B|$). Obviously the same method can also be applied to unbounded spins satisfying estimates weaker than (A.18); the resulting function F will be weaker than (A.22).

It is worth remarking that this use of FKG inequalities, though sufficient for our purposes, is not useful for obtaining *quantitative* estimates near the critical point: the constant in (A.21) blows up as $\alpha \downarrow 0$.

(5) *Reflection positivity* ($|A| = 1$ or $|B| = 1$): For $|i|$ sufficiently large, there exists a hyperplane parallel to one of the coordinate axes which separates the set $\operatorname{supp} A$ from the point i. By translation, reflection, and coordinate-interchange invariance, we can assume that $i_1 \geqslant 0$, $i_2 = \cdots = i_d = 0$ and $\operatorname{supp} A \subset \{ j : j_1 \leqslant 0 \}$. Then by reflection positivity and the Schwarz inequality,

$$\begin{aligned} |\langle \varphi^A; \varphi_i \rangle| &= |\langle (\varphi^A - \langle \varphi^A \rangle)(\varphi_i - \langle \varphi_i \rangle) \rangle| \\ &\leqslant \langle \varphi^A; \varphi^{\theta A} \rangle^{1/2} \langle \varphi_i; \varphi_{\theta i} \rangle^{1/2} \\ &= c_A \langle \varphi_0; \varphi_{2i} \rangle^{1/2} \end{aligned} \tag{A.23}$$

where θ is reflection in the hyperplane $\{ j_1 = 0 \}$ and c_A depends on A (and the state) but not on i. If the two-point function decays suitably rapidly, (A.23) is summable over i. This argument is due to Bricmont *et al.*[65]

It is worth remarking that while the above arguments are applicable to very general perturbations φ^B, the apparent generality is deceptive: since (A.8) requires that the bound hold for *all* λ in the interval $[0, \lambda_0]$, it is necessary that the requisite correlation inequalities hold not only for $\Phi^{(0)}$ but also for the perturbed interaction $\Phi^{(\lambda)}$. In practice, this usually restricts the perturbation to be either linear ($|B| = 1$), quadratic ($|B| = 2$), or affecting only a single site ($|\operatorname{supp} B| = 1$). Likewise, the unperturbed interaction

$\Phi^{(0)}$ must usually be a ferromagnetic pair interaction with a possible magnetic field.

In order to verify (A.11), we now need to obtain bounds on the two-point function (and perhaps also on the magnetization) which are uniform in the volume Λ. This also can often be done by correlation inequalities:

(1) For zero boundary conditions ($b = 0$), each $\langle \varphi^A \rangle_\Lambda$ increases with Λ, by GKS II.[26] Hence $\langle \varphi^A \rangle_\Lambda$ is bounded above by $\langle \varphi^A \rangle_\infty$. This is useful for $J < J_c$.

(2) For + boundary conditions, $\langle \varphi_j ; \varphi_k \rangle_\Lambda$ increases with Λ, by GHS. Hence $\langle \varphi_j ; \varphi_k \rangle_\Lambda$ is bounded above by $\langle \varphi_j ; \varphi_k \rangle_\infty$. This is useful for all J (at least for bounded spins).

We are now prepared to deduce the specific fluctuation–dissipation relations which are needed in the present paper. To differentiate the magnetization, set $f_\Lambda = \varphi_0$ for all Λ; then $F(\lambda) = M(\lambda)$. It follows from the foregoing that the derivative (resp. one-sided derivative) $\partial M / \partial H$ is given by

$$\chi = \left. \frac{\partial M}{\partial H} \right|_{H = H_0} = \sum_i \langle \varphi_0 ; \varphi_i \rangle \tag{A.24}$$

provided that the sum on the right is bounded for H in a neighborhood (resp. one-sided neighborhood) of H_0 (uniformly in the volume Λ). Moreover, as noted above, the uniformity in the volume Λ follows immediately from the corresponding bound on the *infinite-volume* two-point function, if we assume + b.c. (for $H \geqslant 0$), bounded spins, and the GHS inequality. Equation (3.2) for $\partial M / \partial J$ can be derived by similar means; the sum has to be bounded for J in a neighborhood of J_0.

To differentiate the internal energy, set $f_\Lambda = (1/2) \sum_{k \in \mathfrak{L}} J_{0k} \varphi_0 \varphi_k$ for all Λ; then $F(\lambda) = U(\lambda)$. The formulas for $\partial U / \partial H$ [Eq. (3.6)] and $\partial U / \partial J$ [sentence following (2.8)] can be derived under suitable hypotheses similar to the above.

To differentiate χ [sentence following (4.1); also Eq. (5.1)] is a somewhat more subtle matter. If we were to take

$$f_\Lambda = \sum_{i \in \mathfrak{L}} (\varphi_0 - \langle \varphi_0 \rangle_\infty)(\varphi_i - \langle \varphi_i \rangle_\infty) \tag{A.25}$$

for all Λ, then for *finite* volume Λ the sum over i in $\langle f_\Lambda \rangle_\Lambda$ or $\langle f_\Lambda ; \Phi'_X \rangle_\Lambda$ might diverge owing to the contributions $i \notin \Lambda$.[18] Instead, it is more natural to take

$$f_\Lambda^{(1)} = \sum_{i \in \Lambda} (\varphi_0 - \langle \varphi_0 \rangle_\infty)(\varphi_i - \langle \varphi_i \rangle_\infty) \tag{A.26}$$

[18] Unless we happened to take exactly the boundary condition $b_i = \langle \varphi_i \rangle_\infty \equiv M$. Of course, for $J < J_c$ and $H = 0$ we can do just that: take $b_i = M = 0$ (zero b.c.). But if $M \neq 0$, it is inconvenient to take $b_i = M$: we would have to renounce those arguments based on correlation inequalities (see above) which are valid only for zero or + b.c.

or

$$f_\Lambda^{(2)} = \sum_{i \in \Lambda} (\varphi_0 - \langle\varphi_0\rangle_\Lambda)(\varphi_i - \langle\varphi_i\rangle_\Lambda) \tag{A.27}$$

Indeed,

$$\langle f_\Lambda^{(2)}\rangle_\Lambda = \sum_{i \in \Lambda} \langle\varphi_0;\varphi_i\rangle_\Lambda = \frac{\partial}{\partial H}\langle\varphi_0\rangle_\Lambda \tag{A.28}$$

and the fluctuation–dissipation relation (A.24) assures us (under suitable hypotheses) that this approaches χ as $\Lambda\uparrow\mathfrak{L}$. Hence, if we take $f_\Lambda = f_\Lambda^{(2)}$, we have $F(\lambda) = \chi(\lambda)$. Moreover,

$$\langle f_\Lambda^{(2)};\Phi'_X\rangle_\Lambda = \sum_{i \in \Lambda} \langle\varphi_0;\varphi_i;\Phi'_X\rangle_\Lambda \tag{A.29}$$

In order to satisfy (A.8), we need a bound on (A.29) that is uniform in Λ and λ and summable over X. To derive such a bound, we use the FKG inequalities as above. Assume first that the spins are bounded in absolute value by K. Then

$$\begin{aligned}|\langle\varphi^A;\varphi^{B+i};\varphi^{C+j}\rangle| &= |\langle\varphi^A\varphi^{B+i};\varphi^{C+j}\rangle - \langle\varphi^A\rangle\langle\varphi^{B+i};\varphi^{C+j}\rangle \\ &\quad - \langle\varphi^{B+i}\rangle\langle\varphi^A;\varphi^{C+j}\rangle| \\ &\leqslant 2K^{|A|+|B|+|C|-2}\left[\sum_{\substack{k\in\operatorname{supp}A\\ l\in\operatorname{supp}C}}\langle\varphi_k;\varphi_{l+j}\rangle + \sum_{\substack{k\in\operatorname{supp}B\\ l\in\operatorname{supp}C}}\langle\varphi_{k+i};\varphi_{l+j}\rangle\right]\end{aligned} \tag{A.30}$$

But analogous inequalities hold also for the other two permutations of A, B, C; hence the left side of (A.30) is bounded by the *minimum* of these three permutations. Moreover, this minimum is summable over i and j (if the two-point function decays fast enough so that $\xi_d < \infty$): the essential reason is that if one of the distances $|i|$, $|j|$, and $|i-j|$ is large, then in fact at least *two* of them are large; this means that at least one of the three permutations of the right side of (A.30) will be small. Let us show this in detail for the simple case where $|A| = |B| = |C| = 1$. We have

$$\begin{aligned}&\sum_{i,j}|\langle\varphi_0;\varphi_i;\varphi_j\rangle| \\ &\quad\leqslant 2K\sum_{i,j}\min\left[G(i)+G(j), G(i)+G(i-j), G(j)+G(i-j)\right] \\ &\quad\leqslant 12K\sum_{|i|\geqslant|j|\geqslant|i-j|}\left[G(i)+G(j)\right] \\ &\quad\leqslant 12K\left[\sum_i\sum_{\substack{j\\|j|\leqslant|i|}}G(i)+\sum_j\sum_{\substack{i\\|i|\leqslant 2|j|}}G(j)\right] \\ &\quad\leqslant (c_1 + c_2\xi_d^d)\chi\end{aligned} \tag{A.31}$$

where we have written $G(i) = \langle\varphi_0;\varphi_i\rangle$. This handles $\partial\chi/\partial H$; similar but slightly messier arguments handle $\partial\chi/\partial J$. Unbounded spins can be be handled by the superstability argument noted above.

Finally, it is worth noting an argument due to Bricmont[67] which shows that (A.24) holds even *at* the critical point (in the sense, of course, that both sides are $+\infty$). To see this, note first that we *always* have

$$\left.\frac{\partial M}{\partial H}\right|_{H=H_0+} \geqslant \sum_i \langle \varphi_0; \varphi_i \rangle_{H_0} \tag{A.32}$$

for $H_0 \geqslant 0$ (and, say, + b.c.); this is an easy argument using GKS II.[27] But the right side of (A.32) dominates the same sum for $H > H_0$, by GHS. Hence, if the right side of (A.32) is finite, we have precisely the uniform bound required to deduce (A.24) via Proposition A.1. On the other hand, if the right side of (A.32) equals $+\infty$, then so does the left side!

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